AIAA JOURNAL
Vol. 41, No. 7, July 2003

Damage Assessment of a Nonlinear Space Antenna Structure Using Quasi-Linearization Methods

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Health monitoring of structural systems for the detection of damage is a critical practice in determining their safety and service life. A large amount of research has been done in applying damage detection analysis to linear models of structures. Although many real structures exhibit nonlinear behavior before or as a result of damage, there are currently few tools to apply damage detection analysis to structures that are modeled by nonlinear differential equations. The objective of this study is to initiate a facet of damage detection analysis applied to such nonlinear models of structures. Given a parameterized ordinary differential equation model of a structural system (linear or nonlinear), parameter estimation can serve as an assessment of possible or known damage to the structure. A numerical analysis of estimating parameters in a nonlinear structural model is investigated by applying a quasi-linearization method. The analysis is successfully performed on a nonlinear eight-degree-of-freedom space antenna model.

Nomenclature

$\boldsymbol{a}$	=	unknown parameter vector in equations of motion
$\boldsymbol{a}_0$	=	true values for the unknown vector $\boldsymbol{a}$
$\boldsymbol{b}$	=	vector from cost functional minimization
$\boldsymbol{C}$	=	augmented vector of initial conditions for the state vector $\boldsymbol{X}(t)$
$\boldsymbol{c}$	=	initial condition for the state vector $\boldsymbol{x}(t)$, that is, $\boldsymbol{x}(0)$
$\boldsymbol{D}$	=	matrix from cost functional minimization
$\boldsymbol{e}(t)$	=	error between observed and estimated state vector signals
$\boldsymbol{F}(\boldsymbol{X}, \boldsymbol{f})$	=	nonlinear differential equations for modified Kabe model[1]
$\boldsymbol{f}$	=	applied forcing vector in equations of motion
$\boldsymbol{g}(\boldsymbol{x})$	=	general autonomous nonlinear differential equation function
$\boldsymbol{H}_i^{(n)}(t)$	=	homogeneous part i of the recursive vector $\boldsymbol{x}^{(n)}(t)$
$\boldsymbol{J}(\boldsymbol{x}^{(n)})$	=	Jacobian matrix of $\boldsymbol{g}(\boldsymbol{x})$ at $\boldsymbol{x}^{(n)}(t)$
$\boldsymbol{K}$	=	structural symmetric stiffness matrix
k_i	=	stiffness coefficient i
k_0	=	unknown stiffness coefficient
$\boldsymbol{M}$	=	structural diagonal mass matrix
m	=	number of discrete measurements of observed signals
m_i	=	nodal mass i
N_a	=	dimension of the unknown parameter vector $\boldsymbol{a}$
N_x	=	dimension of the state vector $\boldsymbol{x}(t)$
N_{xa}	=	combined dimension of state and parameter vectors
$\boldsymbol{P}^{(n)}(t)$	=	particular part of the recursive vector $\boldsymbol{x}^{(n)}(t)$
$\boldsymbol{r}(t)$	=	residual between observed and estimated position signals
T	=	length of observation time interval
$\boldsymbol{X}(t)$	=	augmented state vector for modified Kabe model
$\boldsymbol{x}, \ddot{\boldsymbol{x}}$	=	vectors of nodal (mass) displacements, accelerations
$\boldsymbol{x}^{(n)}(t)$	=	recursive estimation of the state vector $\boldsymbol{x}(t)$
$\boldsymbol{x}(t)$	=	general continuous state vector
α	=	nonlinear (cubic) stiffness parameter
ϕ	=	least-squares cost functional

Received 30 May 2000; revision received 1 February 2003; accepted for publication 4 February 2003.

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I. Introduction

TO detect damage in a structural system with nonlinearities, a reliable model of the system dynamics must be obtained. Fault generally refers to a reduction in the stiffness values that parameterize the model. For a given model of a structural system (linear or nonlinear), the following subsequent steps are involved in the damage detection process:

1) Detect a change in expected system operating conditions.
2) Locate where the damage occurred in the system.
3) Estimate the reduction in stiffness in the damaged element.
4) Perform a structural analysis, that is, finite element, and determine whether the damaged system is safe by design requirements, or estimate the remaining life.

Step 3 is referred to as damage assessment. The literature related to structural systems deals with model identification and damage detection in linear systems or strictly model identification of systems with nonlinearities, with far more literature in the first area.

The identification and damage detection of ground and space structural systems with linear models has been extensively treated with modal tools. Kabe[1] used measured mode data to identify and adjust the stiffness matrix in the modeling of a severe test-case space antenna structure. Potential damage can be located and estimated using a weighted sensitivity analysis that accommodates mass and stiffness uncertainty, as investigated by Ricles and Kosmatka.[2] Papadopoulos and Garcia[3] used modal information and applied a statistical approach to identify structural damage. Modal tools parametrically identify linear structures while keeping the physically based model structure intact. However, these tools do not have an analog in nonlinear systems. Natke and Yao gave a brief investigation of system identification (SI) approaches for structural damage evaluation in linear civil engineering structures.[4]

Parameter estimation techniques for identification of models with nonlinearities generally fit under the umbrella of SI. This study examines a parameter estimation method on nonlinear models that are linearly parameterized. SI techniques are concerned with deriving mathematical (dynamic) models of systems based on observed data from the systems.[5] Parameter estimation tools are required to complete the derivation when parametric (vs nonparametric) modeling is employed. The application of SI tools to the identification of nonlinear structures dates back to a paper by Distefano and Rath in 1975 (Ref. 6). In their milestone paper, parameters associated with a numerically generated nonlinear response were identified for one-degree-of-freedom structural systems subject to seismic conditions. Nayfeh introduced a self-contained perturbation approach that proposes experimental techniques of identifying models and their parameters in low-degree-of-freedom dynamic systems that contain smooth nonlinearities.[7] Imai et al. examined parametric identification of lower-order linear and nonlinear structural systems using least-squares, maximum likelihood, and extended Kalman filtering (see Ref. 8). The issue of identifying model parameters in higher-order nonlinear ordinary differential equation models has yet to be dealt with.

This study extends a parameter estimation technique to assess the damage in a given nonlinear structural model, where the nonlinearity is a result of damage or initial structural configuration. Steps 1 and 2 of the damage detection process are assumed known, or can be determined by other techniques. A quasi-linearization method is derived as the tool for parameter estimation and applied to an eight-degree-of-freedom nonlinear model of a space antenna. By construction, the method can serve equally as a parameter identification tool amenable to higher-dimensional models. A discussion of extending the method to the more general damage detection problem, that is, when steps 1 and 2 are not assumed known, is given. Numerical results display the effectiveness of the method in assessing damage under various loading and nonlinearity conditions.

II. Analytical Model of a Nonlinear Space Antenna Structure

Figure 1 shows an analytical test model of a flexible space antenna as given by Kabe.[1] As Fig. 1 shows, the antenna is represented by eight interconnected masses in series, that is, all masses move translationally and colinearly, on a frictionless surface. Each mass has one degree of freedom, and 14 springs connect the masses to each other or to a ground. Because the relative stiffness magnitude range is large (from 1.5 to 1000), this structure represents a severe test case. There are three unique lumped mass values and six unique stiffness coefficients. All units are normalized, and the dimensionless equations of motion for this eight-degree-of-freedom statically coupled system are given as

$$M\ddot{x} + Kx = f \tag{1}$$

where the diagonal mass matrix is given by

$$M = \begin{bmatrix} \ddots & & \\ & m_i & \\ & & \ddots \end{bmatrix} \tag{2}$$

the coupled stiffness matrix is given by

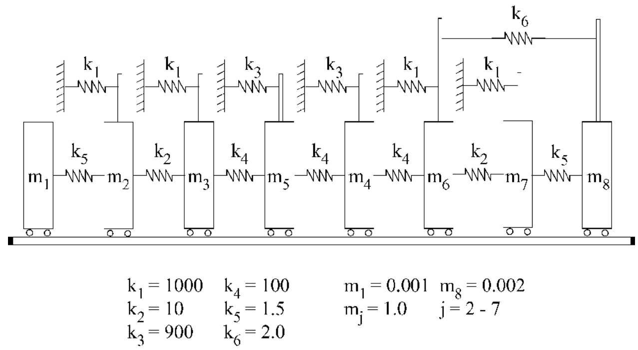

Fig. 1 Kabe model of space antenna.

and the vectors of the nodal displacements and applied forces are given by

$$x = [\,x_1, x_2, \ldots, x_8\,]^T, \qquad f = [\,f_1, f_2, \ldots, f_8\,]^T \tag{4}$$

This model is linear and is made nonlinear here, in the interest of this study. The spring k_1 at m_6 is replaced by a nonlinear Duffing-type spring, henceforth k_1^*, to simulate a spring that hardens/softens as a result of damage or that is initially nonlinear. The mathematical definition is given as

$$k_1^* = k_0\left(1 + \alpha x_6^2\right) \tag{5}$$

This spring location was chosen because it affects the greatest number of modes in the undamaged linear structure, in which case k_1^* is equal to k_1. The added nonlinearity generates a Duffing-type equation

$$m_6\ddot{x}_6 + k_0 x_6\left(1 + \alpha x_6^2\right) = \tilde{f}_6 \tag{6}$$

where

$$\tilde{f}_6 = k_2(x_7 - x_6) + k_4(x_4 - x_6) + k_6(x_8 - x_6) + f_6 \tag{7}$$

All other equations are retained in their original linear form. The stiffnesses k_0 and $k_0\alpha$ at m_6 are assumed unknown constants and are to be estimated. For added complexity, the adjacent k_4 at m_6 is also assumed unknown and requires estimation. All other stiffness values and all mass values are known. It is, henceforth, assumed that the units in the equations are normalized so that the equations to follow remain dimensionless, as is done in Ref. 9.

III. Quasi-Linearization

Quasi-linearization was developed as a numerical tool for solving problems defined by nonlinear differential equations and has been extended to identification problems.[9,10] Under certain assumptions, the technique can successfully reduce a nonlinear optimization problem to a succession of operations involving the numerical solution of linear differential and algebraic equations. First, the equations of motion are linearized about a nominal trajectory into the form of a sequence of equations. A linear solution to these equations is formulated and parameterized in terms of the unknown constants.

$$K = \begin{bmatrix} k_5 & -k_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -k_5 & k_1+k_2+k_5 & -k_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -k_2 & k_1+k_2+k_4 & 0 & -k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_3+2k_4 & -k_4 & -k_4 & 0 & 0 \\ 0 & 0 & -k_4 & -k_4 & k_3+2k_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_4 & 0 & k_1+k_2+k_4+k_6 & -k_2 & -k_6 \\ 0 & 0 & 0 & 0 & 0 & -k_2 & k_1+k_2+k_5 & -k_5 \\ 0 & 0 & 0 & 0 & 0 & -k_6 & -k_5 & k_5+k_6 \end{bmatrix} \tag{3}$$

Second, the resulting sequence of linear differential equations are solved to generate a linear solution, composed of particular and homogeneous parts. Third, given an applied loading and resulting response (displacement) data from the original equations of motion, a cost function is minimized in a least-squares sense, yielding a set of linear algebraic equations. The solution of these equations generates the next estimate for the unknown parameters. Steps 2 and 3 are repeated until the parameters converge to their true values, as discussed in the following sections.

A. Step 1: Linearization of Equations and Linear Solution Form

A set of autonomous nonlinear first-order differential equations can be expressed in the form

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{g}(\boldsymbol{x}(t)), \qquad \boldsymbol{x}(0) = \boldsymbol{c}, \qquad t \geq 0 \tag{8}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^{N_x}$, $\dot{\boldsymbol{x}}(t)$ indicates derivative of $\boldsymbol{x}(t)$ with respect to time, and $\boldsymbol{c} \in \mathbb{R}^{N_x}$ represents the initial condition vector. Assume that $\boldsymbol{g}$ is continuous in $\boldsymbol{x}$ and t and has a bounded continuous Hessian (second partial derivatives with respect to $\boldsymbol{x}$), for all $\boldsymbol{x}$ and t over a given interval of time. The first-order Taylor series expansion of Eq. (8) around a nominal trajectory $\boldsymbol{x}^{(0)}(t)$ is the linear differential equation

$$\frac{\mathrm{d}\boldsymbol{x}^{(1)}}{\mathrm{d}t} = \boldsymbol{g}(\boldsymbol{x}^{(0)}) + \frac{\partial \boldsymbol{g}(\boldsymbol{x}^{(0)})}{\partial \boldsymbol{x}}(\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(0)}), \qquad \boldsymbol{x}^{(1)}(0) = \boldsymbol{c} \tag{9}$$

where $\boldsymbol{x}^{(0)}(t)$ is an approximate solution to Eq. (8). Replacing 1 by n and 0 by $n-1$ in Eq. (9) gives general recurrence equations that yield a sequence of continuous solutions.

According to Bellman and Roth,[9] the success of convergence of the recurrence solutions to the true solution depends principally on selecting $\boldsymbol{x}^{(0)}(t)$ sufficiently close to the true solution. This is analogous to the dependence of Newton–Raphson schemes on the initial guess for success in root finding. Under the bounded Hessian assumption, the sequence solutions $\boldsymbol{x}^{(n)}(t)$, $n = 1, 2, \ldots,$ are uniformly bounded. Furthermore, the sequence of solutions can be proven to converge quadratically to the solutions of the original equations of motion, if they converge at all. Now, $\boldsymbol{g}$ in Eq. (8) is rewritten to include dependence on unknown constant linear parameters. By the collection of the unknowns into the vector $\boldsymbol{a}$, Eq. (8) is augmented as

$$\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ \dot{\boldsymbol{a}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} \boldsymbol{x}(0) \\ \boldsymbol{a}(0) \end{bmatrix} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{a}_0 \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{a}(t) \end{bmatrix} \in \mathbb{R}^{N_x + N_a} \tag{10}$$

The unknown parameter vector $\boldsymbol{a}$, of dimension N_a, is equal to its unknown initial condition $\boldsymbol{a}_0$ for all time.

The vector $\boldsymbol{x}^{(n)}(t) \in \mathbb{R}^{N_{xa}}$ that evolves according to Eq. (9) now represents the recursive augmented state, where, dimensionally, $N_{xa} = N_x + N_a$. A solution form to the augmented set of recursive linear differential state equations is assumed as a sum of particular and homogeneous parts (by superposition) as

$$\boldsymbol{x}^{(n)}(t) = \boldsymbol{P}^{(n)}(t) + \sum_{i=1}^{N_a} a_i\, \boldsymbol{H}^{(n)}_{N_x+i}(t) \tag{11}$$

where $\boldsymbol{P}^{(n)}$ is the particular part, $\boldsymbol{H}^{(n)}_{N_x+i}$ make up the homogeneous parts, the initial conditions are

$$\boldsymbol{P}^{(n)}(0) \stackrel{\triangle}{=} \boldsymbol{P_0} = [\,\boldsymbol{c}^T, \boldsymbol{0}^T_{N_a}]^T, \qquad \boldsymbol{0}^T_{N_a} = [\ldots 0, \ldots] \in \mathbb{R}^{N_a} \tag{12}$$

$$\boldsymbol{H}^{(n)}_{N_x+i}(0) \stackrel{\triangle}{=} \boldsymbol{H_0} = [\ldots, \delta_{j,N_x+i}, \ldots]^T, \qquad j = 1, \ldots, N_{xa}$$

$$i = 1, \ldots, N_a \tag{13}$$

and $\delta_{i,j}$ is the Kronecker delta function. Section III.B gives the equations that define how $\boldsymbol{P}^{(n)}$, $\boldsymbol{H}^{(n)}_{N_x+1}, \ldots, \boldsymbol{H}^{(n)}_{N_{xa}}$, evolve. The goal of this recursive technique is to get the estimated parameter vector $\boldsymbol{a} \in \mathbb{R}^{N_a}$ to converge to $\boldsymbol{a_0}$, thereby matching the initial conditions of Eq. (11) with the initial conditions of the augmented system (10). The solution form (11) assumes that all initial conditions for the equations of motion, that is, the $\boldsymbol{c}$ vector, are known. However, the method can be easily extended to identify any unknown initial conditions as well.

To begin iterations, a nominal trajectory, that is, an initial approximate solution to system (10), is required. As stated, the success of convergence of the approximate solutions to the observed solutions, and, therefore, the success of convergence of the parameter estimates to the true parameter values, depends on this initial approximation. In numerical applications, a simple and efficient way to obtain a first approximate solution $\boldsymbol{x}^{(0)}(t)$ is to make a reasonable guess at the unknown parameter values and integrate the original equations of motion.

B. Step 2: Generation of Recursive Solution Form

Plugging Eq. (11) into Eq. (9) yields a recursive set of first-order linear differential equations, where the particular part of the solution evolves according to

$$\frac{\mathrm{d}\boldsymbol{P}^{(n)}}{\mathrm{d}t} = \boldsymbol{g}(\boldsymbol{x}^{(n-1)}) + \boldsymbol{J}(\boldsymbol{x}^{(n-1)})(\boldsymbol{P}^{(n)} - \boldsymbol{x}^{(n-1)})$$

$$\boldsymbol{P}^{(n)}(0) = \boldsymbol{P_0} \tag{14}$$

the homogeneous part of the solution evolves according to

$$\frac{\mathrm{d}\boldsymbol{H}^{(n)}_{N_x+i}}{\mathrm{d}t} = \boldsymbol{J}(\boldsymbol{x}^{(n-1)})\boldsymbol{H}^{(n)}_{N_x+i}, \qquad \boldsymbol{H}^{(n)}_{N_x+i}(0) = \boldsymbol{H_0}$$

$$i = 1, \ldots, N_a \tag{15}$$

and the Jacobian matrix is given by

$$\boldsymbol{J}(\boldsymbol{x}^{(n-1)}) = \frac{\partial \boldsymbol{g}(\boldsymbol{x}^{(n-1)})}{\partial(\boldsymbol{x}, \boldsymbol{a})} \in \mathbb{R}^{N_{xa} \times N_{xa}}, \qquad n = 1, 2, \ldots \tag{16}$$

C. Step 3: Minimization of Cost Function and Estimate Generation

It is assumed that $\boldsymbol{x}(t)$ is observed over a finite interval $t \in [0, T]$. A least-squares method provides a cost function to be minimized at each iteration, where the cost function is

$$\phi = \int_0^T \left\| \boldsymbol{P}^{(n)}(t) + \sum_{i=1}^{N_a} a_i \boldsymbol{H}^{(n)}_{N_x+i}(t) - \boldsymbol{x}(t) \right\|^2 \mathrm{d}t \tag{17}$$

Note that Eq. (17) is the squared $\boldsymbol{L}_2(0, T)$ norm of the time-varying error vector, where the error is between the linearized solution $\boldsymbol{x}^{(n)}(t)$ and the observed solution $\boldsymbol{x}(t)$. As will be seen in the next section, the entire state vector $\boldsymbol{x}(t)$ is not required to generate estimates. Minimizing a least-squares function such as this to obtain a best-fit set of parameters is the most common method of parameter estimation. It is implied in Eq. (17) that errors superimposed on the observed data $\boldsymbol{x}(t)$ are uniformly distributed throughout the time interval, that is, the weighting function in Eq. (17) is identity for all time in the interval. The minimization of ϕ with respect to the estimated parameter vector $\boldsymbol{a}$ yields a system of linear algebraic equations

$$\frac{\partial \phi}{\partial a_i} = 0, \; i = 1, \ldots, N_a \Longrightarrow \boldsymbol{Da} = \boldsymbol{b} \tag{18}$$

where

$$D_{ji} = \int_0^T \boldsymbol{H}^{(n)}_{N_x+j}(t)^T \boldsymbol{H}^{(n)}_{N_x+i}(t)\, \mathrm{d}t, \qquad \boldsymbol{D} \in \mathbb{R}^{N_a \times N_a} \tag{19}$$

$$b_j = \int_0^T \left[\boldsymbol{x}(t) - \boldsymbol{P}^{(n)}(t)\right]^T \boldsymbol{H}^{(n)}_{N_x+j}(t)\, \mathrm{d}t, \qquad \boldsymbol{b} \in \mathbb{R}^{N_a} \tag{20}$$

$\boldsymbol{D}$ is a symmetric matrix and the solution to the estimation problem at each iteration becomes $\boldsymbol{a} = \boldsymbol{D}^{-1}\boldsymbol{b}$ for nonsingular $\boldsymbol{D}$. For discrete observations, the integrals in the minimization are replaced by summations.

IV. Application to Modified Kabe Model

As stated, the estimation is performed under the assumption that all constants in Eq. (1), modified by Eq. (6), are undamaged (known), except that the spring k_1^* at m_6 is damaged (unknown) and possibly nonlinear, which depends on k_0 and α. Again, the adjacent k_4 at m_6 is also assumed to be damaged (unknown). Before linearization, we write the modified equations of motion in state-space form as

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}, \boldsymbol{f}), \qquad \boldsymbol{X}(0) = \boldsymbol{C}, \qquad t \geq 0 \tag{21}$$

where

$$\boldsymbol{X} = \left[x_1, \dot{x}_1, \ldots, x_8, \dot{x}_8, \boldsymbol{a}^T \right]^T, \qquad \boldsymbol{a} = [a_1, a_2, a_3]^T \tag{22}$$

$$\boldsymbol{f} = [f_1, \ldots, f_8]^T, \qquad \boldsymbol{F} = [\dot{x}_1, \ldots, \ddot{x}_8, 0, 0, 0]^T \tag{23}$$

$$\boldsymbol{C} = [x_1(0), \ldots, \dot{x}_8(0), \boldsymbol{a}_0]^T = \left[\boldsymbol{c}^T, \boldsymbol{a}_0^T\right]^T \tag{24}$$

Dimensionally, it should be clear that $\boldsymbol{X}, \boldsymbol{F} \in \mathbb{R}^{19}$ and $\boldsymbol{a}, \boldsymbol{a}_0 \in \mathbb{R}^3$. The unknown parameter vector $\boldsymbol{a}$ is equal to its unknown initial condition $\boldsymbol{a}_0 = [k_4, k_0, k_0\alpha]^T$ for all time in an observed interval.

The linear recursive approximate solutions for this system are given by

$$\boldsymbol{X}^{(n)}(t) = \boldsymbol{P}^{(n)}(t) + \sum_{k=1}^{3} a_k \boldsymbol{H}_{16+k}^{(n)}(t), \qquad n = 1, 2, \ldots \tag{25}$$

where

$$\frac{\mathrm{d}\boldsymbol{P}^{(n)}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}^{(n-1)}) + \boldsymbol{J}(\boldsymbol{X}^{(n-1)})(\boldsymbol{P}^{(n)} - \boldsymbol{X}^{(n-1)})$$

$$\boldsymbol{P}^{(n)}(0) = [\boldsymbol{c}^T, 0, 0, 0]^T \tag{26}$$

$$\frac{\mathrm{d}\boldsymbol{H}_{16+k}^{(n)}}{\mathrm{d}t} = \boldsymbol{J}(\boldsymbol{X}^{(n-1)})\boldsymbol{H}_{16+k}^{(n)}, \qquad \boldsymbol{H}_{16+k}^{(n)}(0) = [\ldots, \delta_{j,16+k}, \ldots]^T$$

$$j = 1, \ldots, 19 \tag{27}$$

$$k = 1, 2, 3, \qquad \boldsymbol{J}(\boldsymbol{X}^{(n-1)}) = \frac{\partial \boldsymbol{F}(\boldsymbol{X}^{(n-1)}, \boldsymbol{f})}{\partial \boldsymbol{X}} \qquad \left(J_{ij} = \frac{\partial F_i}{\partial X_j}\right)$$

For m discrete measurements in time of the eight mass displacements, the cost function becomes

$$\phi = \sum_{i=1}^{m} \sum_{j=1}^{8} \left[x_j^{(n)}(t_i) - x_j(t_i)\right]^2 \tag{28}$$

and the minimization follows according to Eqs. (18–20). A guess at the true parameter values is used for $\boldsymbol{a}$, to generate an approximate solution $\boldsymbol{X}^{(0)}(t)$ by integration of the original equations of motion, which, in turn, is used to initiate the iterations. For all iterations that follow the initial guess stage, only the parameter vector $\boldsymbol{a}$ changes, as generated by the minimization of Eq. (28).

V. Extension of Method to General Damage Detection Problem

The analysis thus far has considered the identification of unknown parameters that preexist or result from damage by the use of observed responses that contain the faulty information throughout the measurement. A more realistic situation involves response measurements that contain the occurrence of the fault, that is, the measurements change behavior at some point within the observed time interval due to the presence of a fault. Furthermore, the fault may result in hardening or softening of a structural spring, which may also remain linear or become nonlinear. In this more realistic case, detection and location of the fault must precede any assessment or estimation of the damage.

Quasi-linearization identifies unknown linear parameters and generates an iterated linearized solution to the nonlinear differential equations that describe a structural system's dynamics, given some observed dynamic response. The relationship between the observed and generated responses is given by

$$\boldsymbol{x}(t) = \boldsymbol{x}^{(n)}(t, \boldsymbol{a}) + \boldsymbol{e}(t) = \boldsymbol{P}^{(n)}(t) + \sum_{i=1}^{N_a} a_i \boldsymbol{H}_{N_x+i}^{(n)}(t) + \boldsymbol{e}(t) \tag{29}$$

where $\boldsymbol{x}(t)$ is the observed response and $\boldsymbol{x}^{(n)}(t)$ is the nth iterative response parameterized by the vector $\boldsymbol{a}$. The vector $\boldsymbol{e}(t)$ represents the error between the observed and generated signals. When convergence of the parameters is achieved, the L_2 norm of $\boldsymbol{e}(t)$ is minimized over the observed time interval.

In health monitoring of a system for fault detection, a residual serves as a detection gauge because it carries information that is sensitive in some way to the occurrence of a fault. In structural systems where quasi-linearization is employed, a residual $\boldsymbol{r}(t)$ is given by

$$\boldsymbol{r}(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}^{(n)}(t) \tag{30}$$

where $\hat{\boldsymbol{x}}(t)$ is the currently observed dynamic response, for example, mass position measurements. To generate current linearized responses $\boldsymbol{x}^{(n)}(t)$, the given loading must be known and the initial conditions obtainable. Recall that the method can be constructed to estimate any unknown initial conditions. In the absence of a structural fault, that is, when none of the stiffness values have been compromised, $\boldsymbol{r}(t) = \boldsymbol{e}(t)$, and, accordingly, it remains small. When a fault occurs, $\boldsymbol{r}(t)$ exceeds $\boldsymbol{e}(t)$, and a fault is detected.

For the case in which damage has been detected within a set of data measurements, two categories of problems of location and estimation of the new fault are discussed here. Consider first that the fault affects only those parameters that were estimated earlier by quasi-linearization. Next, and, more likely, the fault affects already known parameters and possibly generates new nonlinear behavior in the system response.

First, consider that the fault location is restricted so that damage affects only those parameters that were estimated earlier. The equations that define the algorithm remain unchanged. The problem then becomes one of detection and assessment. Monitoring the residuals defined by Eq. (30) is effective for detection and reveals the time of occurrence of the fault. The issue of reassessment is more complicated, and several approaches are suggested to accommodate this situation. An appropriate weighting, for example, exponential, can magnify the information containing the postdamage responses in the iterative estimations, and, therefore, the fit parameters will gravitate more toward their faulty values. A simpler approach would be to use only the data that were measured after the occurrence of the fault. The particular part of the recursive equations requires the initial conditions of the state variables, that is, requires the position and velocity measurements at the beginning of the postfault measurements. For the application to the modified Kabe model with mass positions observed, the initial postfault positions can extracted from the measurements, given that any noise on the measurements is not too large. However, the eight velocity values are not likely available. In this case, the recursive equations and linear algebraic equations are reformulated for the case of eight unknown initial velocities. This greatly increases the computational load but eliminates the generation of weighting functions and only requires extension of the already proven quasi-linearization approach. This unknown initial condition approach could accommodate multiple hardening and/or softening faults in time and be applied recursively to the data to estimate the corresponding changes due to damage in the linear and nonlinear structural springs estimated earlier.

Now consider a more difficult and likely situation. When softening or hardening damage occurs in a linear spring whose stiffness was not already estimated, but known, or when damage results in more added nonlinear terms of known form, for example, an added cubic stiffness term, the quasi-linearization approach is still successful but requires redefinition of the iterative linear differential and algebraic equations. When the potential location of damage is known, this reduces the computation requirements because the algorithm is structured to estimate these unknowns. With no knowledge of the location of potential damage, a comparative process where sets of parameters are assumed unknown can be performed. The estimated sets are compared to their known values for discrepancies. This would serve as an alternative to assuming that all parameters are possibly fault sensitive.

VI. Numerical Results

In this section, the application defined in Sec. IV is examined by numerical simulations. Various loadings are applied to the developed nonlinear antenna model. Furthermore, the damage level, (the value of α), the time step of integration, and the initial guesses for the parameters are varied to examine conditions for successful convergence. Conditions of noise contamination on the observed displacement measurements are also examined. All units are normalized, and the model is integrated with a step size of 0.01 s, unless specified otherwise. The simulations were performed in MATLAB (Ver. 5.2) and Simulink (Ver. 2.0).[11]

As Tables 1 and 2 show, for α variations and applied step loads at m_5 of $f_5 = 1000$ and 4000, the two linear parameters (k_4, k_0) converged to three and four significant digits, and the nonlinear parameter $(k_0\alpha)$ converged to two significant digits, all within two iterations. In all tabulated results that follow, converged values are defined as those that do not change to four significant figures after two or more iterations.

Table 3 shows that for an applied step load at m_6 of $f_6 = 1000$, the parameter convergence accuracy deteriorates to failure as α is increased. For the most damaged or nonlinear case, that is, $\alpha = 10$, the iterations remained bounded but never converged to any values. Using even the true values (100, 1000, and 10,000) as an initial guess (not shown Table 3) does not yield convergence. Also, for an initial guess of (1, 1, and 1), all converged cases in Tables 1–3 remained convergent in the same number of iterations (not shown in Table 3).

Table 1 Convergence results for variable α with f_5 = 1000

True values $(k_4, k_0, \alpha k_0)$	Initial guess	First iteration	Second iteration	Converged values
$\begin{bmatrix}100\\1000\\100\end{bmatrix}$	$\begin{bmatrix}50\\1300\\145\end{bmatrix}$	$\begin{bmatrix}100.5\\997.1\\101.4\end{bmatrix}$	$\begin{bmatrix}99.99\\1000\\101.9\end{bmatrix}$	$\begin{bmatrix}99.99\\1000\\101.9\end{bmatrix}$
$\begin{bmatrix}100\\1000\\500\end{bmatrix}$	$\begin{bmatrix}150\\1300\\320\end{bmatrix}$	$\begin{bmatrix}99.59\\997.2\\506.1\end{bmatrix}$	$\begin{bmatrix}99.98\\1000\\509.7\end{bmatrix}$	$\begin{bmatrix}99.98\\1000\\509.7\end{bmatrix}$
$\begin{bmatrix}100\\1000\\2500\end{bmatrix}$	$\begin{bmatrix}150\\750\\1700\end{bmatrix}$	$\begin{bmatrix}99.71\\1002\\2548\end{bmatrix}$	$\begin{bmatrix}100.0\\1000\\2543\end{bmatrix}$	$\begin{bmatrix}100.0\\1000\\2543\end{bmatrix}$
$\begin{bmatrix}100\\1000\\10{,}000\end{bmatrix}$	$\begin{bmatrix}150\\750\\7500\end{bmatrix}$	$\begin{bmatrix}98.80\\996.0\\10{,}520\end{bmatrix}$	$\begin{bmatrix}100.1\\1001\\10{,}130\end{bmatrix}$	$\begin{bmatrix}100.1\\1000\\10{,}140\end{bmatrix}$

Table 2 Convergence results for variable α with f_5 = 4000

True values $(k_4, k_0, \alpha k_0)$	Initial guess	First iteration	Second iteration	Converged values
$\begin{bmatrix}100\\1000\\100\end{bmatrix}$	$\begin{bmatrix}50\\1300\\145\end{bmatrix}$	$\begin{bmatrix}100.5\\998.9\\101.4\end{bmatrix}$	$\begin{bmatrix}100.0\\1000\\101.7\end{bmatrix}$	$\begin{bmatrix}100.0\\1000\\101.7\end{bmatrix}$
$\begin{bmatrix}100\\1000\\500\end{bmatrix}$	$\begin{bmatrix}150\\1300\\320\end{bmatrix}$	$\begin{bmatrix}99.63\\995.7\\507.4\end{bmatrix}$	$\begin{bmatrix}100.1\\1000\\507.3\end{bmatrix}$	$\begin{bmatrix}100.1\\1000\\507.3\end{bmatrix}$
$\begin{bmatrix}100\\1000\\2500\end{bmatrix}$	$\begin{bmatrix}150\\750\\1700\end{bmatrix}$	$\begin{bmatrix}99.37\\999.5\\2574\end{bmatrix}$	$\begin{bmatrix}100.1\\1002\\2532\end{bmatrix}$	$\begin{bmatrix}100.1\\1002\\2532\end{bmatrix}$
$\begin{bmatrix}100\\1000\\10{,}000\end{bmatrix}$	$\begin{bmatrix}150\\750\\7500\end{bmatrix}$	$\begin{bmatrix}99.50\\1001\\10{,}250\end{bmatrix}$	$\begin{bmatrix}100.0\\1002\\10{,}130\end{bmatrix}$	$\begin{bmatrix}100.1\\1002\\10{,}140\end{bmatrix}$

Table 3 Convergence results for variable α with f_6 = 1000

True values $(k_4, k_0, \alpha k_0)$	Initial guess	First iteration	Second iteration	Converged values
$\begin{bmatrix}100\\1000\\100\end{bmatrix}$	$\begin{bmatrix}50\\1300\\145\end{bmatrix}$	$\begin{bmatrix}100.5\\1007\\95.02\end{bmatrix}$	$\begin{bmatrix}100.0\\1003\\99.21\end{bmatrix}$	$\begin{bmatrix}100.0\\1003\\99.21\end{bmatrix}$
$\begin{bmatrix}100\\1000\\500\end{bmatrix}$	$\begin{bmatrix}150\\1300\\320\end{bmatrix}$	$\begin{bmatrix}99.41\\1014\\493.9\end{bmatrix}$	$\begin{bmatrix}101.3\\1003\\497.7\end{bmatrix}$	$\begin{bmatrix}101.3\\1003\\497.7\end{bmatrix}$
$\begin{bmatrix}100\\1000\\2500\end{bmatrix}$	$\begin{bmatrix}150\\750\\1700\end{bmatrix}$	$\begin{bmatrix}99.50\\1005\\2550\end{bmatrix}$	$\begin{bmatrix}55.00\\960.0\\2592\end{bmatrix}$	$\begin{bmatrix}99.83\\1070\\2440\end{bmatrix}$
$\begin{bmatrix}100\\1000\\10{,}000\end{bmatrix}$	$\begin{bmatrix}150\\750\\7500\end{bmatrix}$	$\begin{bmatrix}99.50\\1066\\10{,}140\end{bmatrix}$	$\begin{bmatrix}1536\\11{,}260\\-1024\end{bmatrix}$	——

Table 4 Decrease in time step with f_6 = 1000 and α = 0.1

Time step	True values $(k_4, k_0, \alpha k_0)$	initial guess	First iteration	Second iteration	Converged values
0.01	$\begin{bmatrix}100\\1000\\100\end{bmatrix}$	$\begin{bmatrix}50\\1300\\145\end{bmatrix}$	$\begin{bmatrix}100.5\\1007\\95.02\end{bmatrix}$	$\begin{bmatrix}100.0\\1003\\99.21\end{bmatrix}$	$\begin{bmatrix}100.0\\1003\\99.21\end{bmatrix}$
0.03	$\begin{bmatrix}100\\1000\\100\end{bmatrix}$	$\begin{bmatrix}50\\1300\\145\end{bmatrix}$	$\begin{bmatrix}108.1\\1023\\61.40\end{bmatrix}$	$\begin{bmatrix}98.90\\1002\\103.3\end{bmatrix}$	$\begin{bmatrix}100.3\\1030\\88.90\end{bmatrix}$
0.04	$\begin{bmatrix}100\\1000\\100\end{bmatrix}$	$\begin{bmatrix}50\\1300\\145\end{bmatrix}$	$\begin{bmatrix}58.50\\321.3\\235.9\end{bmatrix}$	$\begin{bmatrix}111.0\\1198\\113.0\end{bmatrix}$	——

Table 5 Variable time step with f_6 input, initial guess = (150, 750, 7500) and α = 10

f_6	Time Step	True values $(k_4, k_0, \alpha k_0)$	First iteration	Second iteration	Converged values
1000	0.01	$\begin{bmatrix}100\\1000\\10{,}000\end{bmatrix}$	$\begin{bmatrix}99.50\\1066\\10{,}140\end{bmatrix}$	$\begin{bmatrix}1536\\11{,}260\\-1024\end{bmatrix}$	——
1000	0.001	$\begin{bmatrix}100\\1000\\10{,}000\end{bmatrix}$	$\begin{bmatrix}99.99\\1066\\10{,}002\end{bmatrix}$	$\begin{bmatrix}100.0\\1001\\99{,}970\end{bmatrix}$	$\begin{bmatrix}100.0\\1002\\99{,}970\end{bmatrix}$
$\bar{f}_6^{a}$	0.01	$\begin{bmatrix}100\\1000\\10{,}000\end{bmatrix}$	$\begin{bmatrix}100.3\\998.0\\9737\end{bmatrix}$	$\begin{bmatrix}99.63\\975.1\\10{,}020\end{bmatrix}$	$\begin{bmatrix}99.63\\974.8\\10{,}010\end{bmatrix}$
$\bar{f}_6^{a}$	0.015	$\begin{bmatrix}100\\1000\\10{,}000\end{bmatrix}$	$\begin{bmatrix}15{,}490\\-12{,}800\\34{,}820\end{bmatrix}$	∞	——

[a]Input, $\bar{f}_6 = 1000(0.35\sin(10t) + 0.3\sin(700t) + 0.25\sin(800t) + 0.2\sin(200t))$

Table 4 shows that convergence of the least nonlinear case of Table 3 is lost as the time step is increased. Table 5 shows that by decreasing the time step by one order of magnitude, that is, from 0.01 to 0.001 s, accurate convergence is obtained in three iterations for the nonconvergent case of Table 3. Table 5 also reveals that by changing the input from a step to a sum of sinusoids, the originally nonconvergent time step converges in three iterations. As with all other cases, the parameters eventually fail to converge as the time step is increased for this sinusoidal input.

Table 6 combines variations in time step as well as in initial guess for a larger input at m_6 of $f_6 = 4000$, while showing computation time per iteration. For the extreme initial guess case in the second row in Table 6, infinity means that the parameters not only did not converge, but they became unbounded. The time-step reduction shown in row three demonstrates that convergence is obtainable for this extreme initial guess at the cost of larger computation time.

Table 6 Variable time step with f_6 = 4000 and (k_4, k_0, and αk_0) = (100, 1000, and 100)

Time step	Initial guess	First iteration	Second iteration	Converged values	Compute time/iteration
0.01	[50, 1300, 145]	[100.4, 1119, 92.42]	[99.96, 1045, 98.21]	[99.96, 1049, 97.97]	2.5 s
0.005	[50, 750, 7500]	[24,580, −1536, 1408]	[3584, 400.5, −512.0]	∞	5.5 s
0.0005	[50, 750, 7500]	[100.0, 1012, 99.05]	[100.0, 1000, 100.0]	[100.0, 1000, 100.0]	50.1 s

Table 7 Measurements (x_3, x_6, and x_8) contaminated by noise for f_5 = 1000 and α = 0.1

Added noise, %	True values (k_4, k_0, αk_0)	Initial guess	First iteration	Second iteration	Converged values
0	[100, 1000, 100]	[50, 1300, 145]	[100.5, 997.1, 101.4]	[99.99, 1000, 101.9]	[99.99, 1000, 101.9]
5	[100, 1000, 100]	[50, 1300, 145]	[98.57, 988.5, 140.5]	[99.99, 1000, 102.3]	[99.96, 1000, 102.1]
10	[100, 1000, 100]	[50, 1300, 145]	[96.51, 983.1, 197.9]	[99.95, 1001, 104.9]	[99.92, 1001, 102.2]
20	[100, 1000, 100]	[50, 1300, 145]	[92.58, 978.9, 334.2]	[98.93, 997.6, 172.1]	[98.82, 1002, 102.6]
50	[100, 1000, 100]	[50, 1300, 145]	[87.90, 989.8, 595.3]	[92.34, 1023, 349.6]	——

The success of convergence depends primarily on the time step of integration and, secondarily, on the type of loading and the magnitude of damage in k_1 present, that is, the magnitude of α. For a small enough time step, all cases considered here converge in one iteration.

The success of the method is also investigated for cases where the system observations are contaminated by noise. Uniformly distributed random noise is added to the measured displacements x_3, x_6, and x_8 at 5, 10, 20, and 50% of the peak value of each of these signals. These cases are considered for the conditions of the first case given in Table 1, and the results are given in Table 7.

A. Discussion of Results

The tabulated results indicate that the time step of integration is the chief factor that determines the success of convergence of the unknown model parameters to their true values. As the time step is decreased, all of the variable conditions of loading, nonlinearity (damage), added noise, and initial guesses eventually result in convergent results. Recall that in the derivation of the quasi-linearization method, an assumption on the differential equations of motion is that the right-hand side remain continuous in time. It is plausible that the dependence of successful convergence on time step is equivalently a dependence of the simulated signals on smoothness in time. To see this, Fig. 2 displays the response of the nonlinear term $x_6(t)^3$ in the original equations of motion under three types of loading. Tables 1–7 reveal that the first two loading types are convergent cases, whereas the third loading is a nonconvergent case. For the third case, the step input $f_6 = 1000$ generates a higher-frequency response in $x_6(t)^3$, and, for the given time step, the signal loses smoothness. Table 5 shows that for a reduction in time step (0.001 s), convergence is achieved. A comparison of $x_6(t)^3$ for these convergent and nonconvergent time steps is given in Fig. 3, showing the improvement in smoothness for the smaller time step.

The effect of continuity in the observed signals, generated by the equations of motion, on the success of convergence is made transparent by examination of the recursive equations. Some of the observed signals become forcing terms in the recursive equations, exposed by computation of the Jacobian. The Jacobian in Eq. (27) yields $x_6(t)$, $x_6(t)^3$, and $x_4(t)$ as the observed signals that serve as inputs to the iterative homogeneous equations (27), appearing in the form

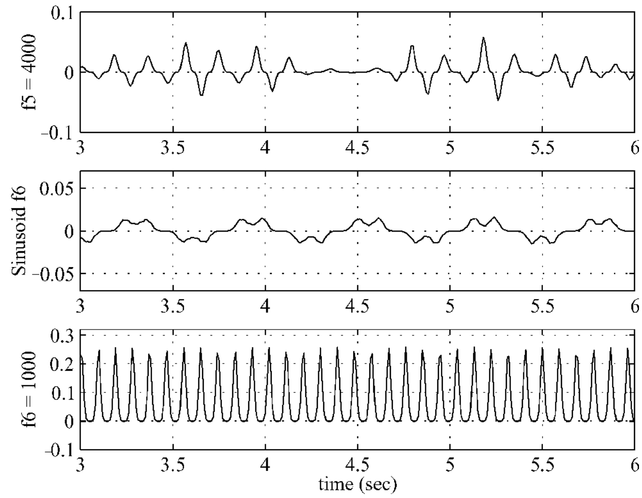

Fig. 2 Measurement of $x_6(t)^3$ for various inputs to relate smoothness and convergence: (α = 10, δ = 0.01).

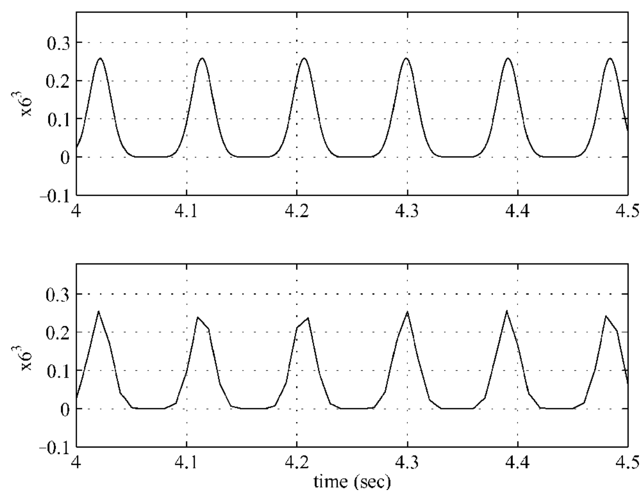

Fig. 3 Smoothness of $x_6(t)^3$ with a smaller time step (case 3 of Fig. 2): ($f6$ = 1000, α = 10).

$$\frac{\mathrm{d}H_{j8}^{(n)}(t)}{\mathrm{d}t} = -\frac{1}{m_4}(k_3 + k_4 + a_1)H_{j7} + \frac{k_4}{m_4}H_{j9} + \frac{a_1}{m_4}H_{j11} + \frac{1}{m_4}(x_6(t) - x_4(t))H_{j17} \tag{31}$$

$$\frac{\mathrm{d}H_{j12}^{(n)}(t)}{\mathrm{d}t} = \frac{a_1}{m_6}H_{j7} - \frac{1}{m_6}(k_2 + k_6 + a_1 + a_2 + 3a_3x_6^2(t))H_{j11} + \frac{k_2}{m_6}H_{j13} + \frac{k_6}{m_6}H_{j15} + \frac{1}{m_6}(x_4(t) - x_6(t))H_{j17} - \frac{x_6(t)}{m_6}H_{j18} - \frac{x_6(t)^3}{m_6}H_{j19} \tag{32}$$

with initial conditions

$$H_{jk}^{(n)}(0) = \delta_{jk}, \qquad j = 17, 18, 19, \qquad k = 1, \ldots, 19 \tag{33}$$

where a_1, a_2, and a_3 are the estimated parameters. These terms are also inputs to the iterative particular equations, along with the model input $\boldsymbol{f}$.

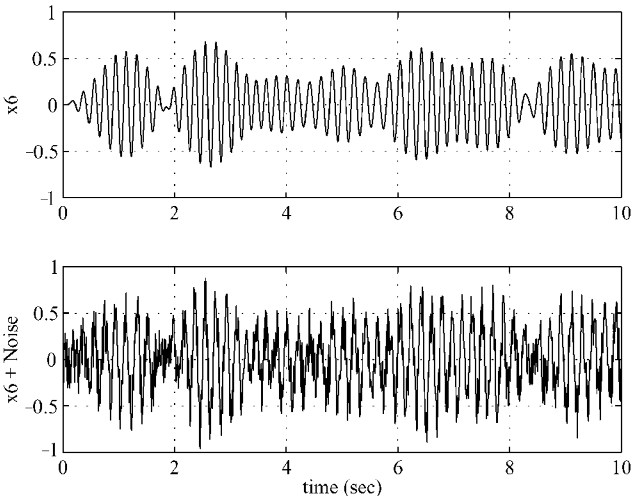

Fig. 4 Loss of smoothness of $x_6(t)$ with 0 and 50% added noise for case 1 in Table 1: ($F5 = 1000$, $\alpha = 0.1$, $\delta = 0.01$ s).

It has been shown (Table 6) that, when a sufficiently small time step is applied, the method displays good robustness with respect to initial guesses on the unknown parameter values. The breakdown of convergence for the case of increasingly added noise (Table 7) is also a case of a loss of smoothness in time for the given time step and, therefore, convergence. To compare the noise-free and noisy measurements, Fig. 4 shows the measured displacement of mass six, $[x_6(t)]$ for 0 and 50% added noise. Clearly, for greater noise content in the observed signals, smoothness is lost, resulting in poorer convergence performance.

To summarize, the tabulated convergence results can be translated into a test of smoothness in time on the observed signals. In particular, the results depend on the smoothness of those signals that become inputs to the recursive equations, made transparent by calculation of the Jacobian.

B. Extension Results

The qualitative description regarding residual generation for detection given in Sec. V can be realized quantitatively as it applies to the nonlinear space antenna structure. For the loading, nonlinearity, and integration conditions of case 1 in Table 1, a softening fault in k_4 from 100 to 50 is induced before any observations. A threshold of ± 0.001 is put on all of the residuals as the signature value, and the fault is detected within 0.02 s. The parameters estimated earlier then serve as the initial guess for a new set of iterations. In this case, the parameters converged to the new true values (50, 1000, and 100) within two iterations. For the same conditions, except that the softening fault at k_4 is induced at $t = 5$ s for 10 s of observed data, difficulty arises because the entire observation data batch can not simply be used for estimation. The use of the entire observed responses for this example illustrates this, because the converged parameter values are (87.92, 960.9, and 421.5) after 10 iterations. Weighting or truncating that data for location and estimation of the fault would have to be incorporated as discussed in Sec. V.

VII. Conclusions

The success of the quasi-linearization method in parametrically identifying unknown parameters that preexist or result from damage in the nonlinear model of the space antenna structure is extensively proven. A wide range of loading, nonlinearity levels, and noise levels in the observed dynamic responses were examined by numerical simulation. Although the method is highly successful for the eighth-order space antenna model, the number of parameters is kept small because the number of differential and linear equations that require solutions for each iteration grows rapidly for a higher number of unknown parameters. Still, as a tool for identifying a limited number of unknown parameters in damaged and, therefore, nonlinear higher-order structural systems, given the model structure and postdamage dynamic response observations, the quasi-linearization approach of parameter estimation is reliable. The approach also shows promise in the detection, location, and assessment of multiple structural faults in such a model.

Acknowledgments

This work was funded by an Achievement Rewards for College Scientists Fellowship, sponsored by the Reuben H. Fleet Foundation. This work is a portion of the M.S. Thesis "Damage Detection of Dynamic Systems with Nonlinearities" by William B. Dunbar, completed in the Applied Mechanics and Engineering Science Department, University of California, San Diego, 1999.

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